Kelvin-Helmholtz instability of relativistic jets – the transition from linear to nonlinear regime

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Abstract: The observed wiggles and knots in astrophysical jets as well as the curvilinear motion of radio emitting features are frequently interpreted as signatures of the Kelvin–Helmholtz (KH) instability (eg. Hardee 1987). We investigate the KH instability of a hydrodynamic jet composed of a relativistic gas, surrounded by a nonrelativistic external medium and moving with a relativistic bulk speed. We show basic nonlinear effects, which become important for a finite amplitude KH modes. Since the KH instability in supersonic jets involves acoustic waves over-reflected on jet boundaries, the basic nonlinear effect relies on the steepening of the acoustic wave fronts, leading to the formation of shocks. It turns our that the shocks appear predominantly in the external nonrelativistic gas, while the internal acoustic waves remain linear for a much longer time. In addition, the external medium "hardens" as soon as the boundary oscillation velocity becomes comparable to the external sound speed. On the other hand, the amplification of internal waves due to the over-reflection is limited by a nonlinearity of the Lorentz γ factor. This implies that the sidereal oscillations of the jet boundary, resulting from the K-H instability, are limited to very small amplitudes comparable to a fraction of the jet radius.

1 Introduction

The Kelvin-Helmholtz (K-H) instability of supersonic jets excites oblique acoustic waves in jets and the ambient medium. Internal acoustic waves are amplified by multiple reflection from jet boundaries with the absolute value of the reflection coefficient $\mathcal{R} = (Z_e - Z_i)/(Z_e + Z_i)$ larger than 1 (Payne and Cohn 1985), where $Z_e = P_e/V_e$, $Z_i = P_i/V_i$ are the complex acoustic impedances and P_i, P_e, V_i, V_e are the pressure and velocity oscillation amplitudes for acoustic waves in the internal and external gases. This effect is called over-reflection and is caused by the Bernouli effect. Acoustic impedance tells us if the medium is 'soft' or 'hard' depending on what pressure oscillation amplitude is necessary to force the unit velocity oscillation amplitude. The contact surface of internal and external gases

undergoes a wavelike deformation, emitting outward propagating acoustic waves in the external medium.

We apply the relativistic equations of hydrodynamics and the relativistic equation of state, with the adiabatic index $\Gamma_i = 4/3$ and internal sound speed $c_{si} = c/\sqrt{3}$, for the jet gas. The external gas is described by the nonrelativistic equations of hydrodynamics and the nonrelativistic equation of state with $\Gamma_e = 5/3$. The boundary conditions ensure that pressure and transversal displacements of the internal and external gases are equal at the jet boundary. In the present considerations we assume 2D slab geometry.

The linearized theory predicts a variety of modes of the K-H instability, classified with respect to their symmetry, longitudinal wavenumber, azimuthal wavenumber and the number of nodes within the jet radius. There is a qualitative correspondence between symmetric (antisymmetric) modes of 2D jets and pinching (helical) modes of cylindrical jets.

The main problem of the linearized theory of the K-H instability is such that it is valid only for infinitesimally small amplitude perturbations. In the next sections we shall investigate some aspects of the nonlinear development of the K-H instability in relativistic, supersonic jets.

2 Nonlinear steepening of acoustic waves

Since the acoustic wave amplitude grows due to the multiple over-reflection, one can expect that sooner or later the nonlinear range will be reached. Observing waves in their direction of propagation we can tentatively apply a one dimensional description. The nonlinear propagation of sound waves in one dimension is analytically described within the theory of simple waves, for which the Riemann invariants allow to specify explicitly nonlinear analytical relations between pressure, density, sound speed and gas velocity (see eg. Landau and Lifshitz (1959) for the classical case and Anile (1989) for the relativistic case). The essential property of nonlinear acoustic waves is such that the propagation speed is dependent on density, what leads to their steepening and formation of shocks.

It is relatively easy to calculate time and the path lengths which is necessary to form shock fronts from an initially sinusoidal wave of a given amplitude. It turns out that for a given mode of K-H instability and perturbation amplitude, the path length measured in the direction perpendicular to the jet axis, necessary for the formation of shocks is an order of magnitude larger in the relativistic jet gas than in the external medium.

Assuming $\gamma = 10$ for the present considerations, the corresponding time necessary to form shocks (measured in the reference frame of external gas) is two orders of magnitude larger in the internal gas due to the additional γ factor . Thus, we expect that for given amplitude of jet boundary oscillations due to K-H instability, the relativistic jet gas should be much more "smooth" than the nonrelativistic external medium or the nonrelativistic jet gas.

3 Nonlinear reaction of external medium

The formalism of simple acoustic waves applied to the nonrelativistic external gas with the adiabatic index $\Gamma_e = 5/3$ leads to the following relation between pressure and velocity (for details see Landau and Lifshitz 1959)

$$p_e = p_{e0}\left(1 + \frac{1}{3}\frac{v_e}{c_{se0}}\right)^5$$

It follows from the above relation that the internal gas feels more and more hard interface, while the perturbation amplitude grows. This means that the external acoustic impedance grows, i.e. one needs much more effort to increase the velocity oscillations of the boundary when the perturbation amplitude is large. The effect becomes important when the jet boundary starts to oscillate with the speed comparable to the external sound speed. A more detailed considerations show, however, that this effect does not limit definitely the displacement amplitude of the contact surface.

4 Reflection of internal acoustic waves

Let us focus our attention on reflection of internal acoustic waves, observed in the reference frame comoving with the unperturbed flow. Since the formation of shocks is very slow in the relativistic internal gas, we shall assume that the fundamental frequency of acoustic waves dominates and the amplitudes of higher harmonics are small.

In the linear range the typical reflection coefficients for the most unstable modes are of the order of 3-10 (eg. for $\gamma = 10$, $\nu = \rho_i/\rho_e = 0.01$). It is obvious, however, that we can not magnify the amplitude of sound waves to an unlimited value during the multiple reflection because of the relativistic limitation of velocity.

Analyzing internal sound waves in the vicinity of a given point on the jet's boundary, we have to take into considerations the sum of velocity perturbations corresponding to the incident and reflected waves. The relativistic velocity limitation can be roughly written for the parallel component of velocity perturbation

$$\left| v_{\parallel}^{(j)+} + v_{\parallel}^{(j)-} \right| < c,$$

where both the incident $v_{\parallel}^{(j)+}$ and reflected $v_{\parallel}^{(j)-}$ components have the same directions and the superscript (j) means the reference frame comoving with the jet velocity. If the parallel velocity oscillations corresponding to the incident wave have already a large amplitude $V_{\parallel}^{(j)+} = c/2$, then the amplitude $V_{\parallel}^{(j)-}$ of the reflected wave has to be smaller than $c/2$. Then we have

$$\mid \mathcal{R} \mid = \frac{\mid V_{\parallel}^{(j)-} \mid}{\mid V_{\parallel}^{(j)+} \mid} < 1$$

This means that there is no more amplification of acoustic waves above a certain amplitude of the incident wave (which is in fact smaller than $c/2$). *Therefore we expect the relativistic saturation of the K-H instability.* This effect is caused by the nonlinearity contained in the Lorentz factor γ in the relativistic Euler equation. The nonlinearity prevents the gas velocity to exceed the speed of light.

Let us estimate the boundary oscillation amplitude corresponding to $V_{\parallel}^{(j)\pm} \sim c/2$. The Lorentz transformation implies

$$V_{\parallel}^{(j)} = \gamma^2 V_{\parallel}^{(e)},$$
$$V_{\perp}^{(j)} = \gamma V_{\perp}^{(e)},$$

where $V_{\parallel}^{(j)}$, $V_{\parallel}^{(e)}$ $V_{\perp}^{(j)}$ and $V_{\perp}^{(e)}$ are perturbations of velocity parallel and perpendicular to the jet axis, measured in the reference frames of jet (j) and the external gas (e).

The linear solution allows to figure out that $\mid V_{\parallel}^{(j)\pm} \mid \sim \mid V_{\perp}^{(j)\pm} \mid$, then the limitation $\mid V_{\parallel}^{(j)\pm} \mid$ to $c/2$ in the jet reference frame implies the corresponding limitation in the reference frame of external gas

$$V_{\parallel}^{(e)\pm} < \frac{c}{2\gamma^2}$$
$$V_{\perp}^{(e)\pm} < \frac{c}{2\gamma}$$

Let us take into account a typical linear solution $k \sim 1/R$, $\omega_r \sim c_{si}/R$, $\omega_i \sim 0.1 c_{si}/R$. These parameters are in fact dependent on the mode of perturbation, but we shall neglect that for simplicity. One can deduce that the maximal displacement amplitude A of the jet boundary is

$$A \sim \frac{V_{\perp}^{(e)\pm}}{|\omega|} \sim \frac{R}{\gamma},$$

where R is the jet radius and ω is the complex frequency. It is apparent that even if the internal gas oscillates with velocity approaching the speed of light (in the jet reference frame), the external boundary oscillates with a very small amplitude of the order of R/γ.

5 Conclusions

1. The K-H instability saturates for very small amplitudes of lateral displacements of jet's boundaries. This saturation is caused by the relativistic limitation of velocity of the jet gas and is additionally supported by the nonlinear reaction of the external medium.
2. The patterns of internal oblique shocks form very slowly in the internal relativistic gas.

3. The results obtained by means of the multiple timescale method, applied to the K-H instability in relativistic jets, lead to the same conclusions (Hanasz, 1995).
4. Our results are consistent with the results of numerical simulations of relativistic jets (Marti et al. 1997, Duncan and Hughes 1994), where the lack of internal jet structure is observed in the case of relativistic equation of state of the jet gas.
5. It seems to be unlikely that the curvilinear motion of superluminal knots in some jets (like 3C 345 or 3C 273) is caused by the Kelvin–Helmholtz instability.

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